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REMARKS ON JOHN DISKS*

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Abstract Let $D \subseteq \overline{R}^2$ be a Jordan domain, $D^* = \overline{R}^2 \setminus \overline{D}$, the exterior of D. In this article, the authors obtained the following results: (1) If D is a John disk, then D is an outer linearly locally connected domain; (2) If D^* is a John disk, then D is an inner linearly locally connected domain; (3) A homeomorphism $f: R^2 \to R^2$ is a quasiconformal mapping if and only if $f(D)$ is a John disk for any John disk $D \subseteq R^2$; and (4) If D is a bounded quasidisk, then D is a John disk, and there exists an unbounded quasidisk which is not a John disk.

Key words John disks, Linearly locally connected domains, Quasiconformal mappings, Quasidisks

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1 Introduction

In this article, we shall adopt the notation and terminology as suggested in article [21], R^2 denotes the 2-dimensional Euclidean space, $\overline{R}^2 = R^2 \cup \{\infty\}$. For $x \in R^2$ and $0 < r < \infty$, let $B^2(x,r) = \{z \in R^2 : \ |z-x| < r\}$, $\overline{B}^2(x,r)$ be the closure of $B^2(x,r)$, $S^1(x,r) = \partial B^2(x,r)$, $B^2(r) = B^2(0,r)$, and $B^2 = B^2(1)$. Suppose that f is a homeomorphism in R^2, let $L(x,f,r) = \max\limits_{|y-x|=r} |f(y)-f(x)|$ and $l(x,f,r) = \min\limits_{|y-x|=r} |f(y)-f(x)|$.

Let D be a Jordan domain in $\overline{R}^2$ and $c \geq 1$ be a constant. We say that D is a $c-$John disk if there exists a point $x_0 \in D$ such that for any point $x_1 \in D$, there exists a rectifiable curve $\gamma \subset D$, which joins x_1 and x_0, satisfying $l(\gamma(x_1,x)) \leq c\mathrm{d}(x,\partial D)$ for any $x \in \gamma$, where $l(\gamma(x_1,x))$ denotes the Euclidean length of the subcurve of γ between x_1 and x, $\mathrm{d}(x,\partial D)$ is the Euclidean distance from x to the boundary ∂D of D. D is called a John disk if D is a c-John disk for some $c \geq 1$.

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John disks were first introduced by F. John in his study [16] of approximation of bi-Lipschitz mappings. Later, O. Martio and J. Sarvas [18], F.W. Gehring, K. Hag and O. Martio [5], S.K. Chua [4], J. Väisälä [22], C.T. McMullen [19], and O.J. Broch [3] studied the John disks extensively. They established many interesting and significant geometric and analytic properties for John disks.

Suppose that D is a Jordan domain in $\overline{R}^2$ and $f:\overline{R}^2\to\overline{R}^2$ is a K-quasiconformal mapping, where $1\le K<+\infty$. D is called a quasidisk if D is the image of the unit disk B^2 under f.

It is well known that quasidisks play a very important role in quasiconformal mapping, complex dynamics, Fuchsian groups, and Teichmüller space theory, see [1, 2, 17, 20].

Next, let $D\subset\overline{R}^2$ be a domain and $c\ge 1$ be a constant. (1) If for any $x_0\in R^2$ and $0<r<+\infty$, points in $D\cap\overline{B}^2(x_0,r)$ can be joined by curves in $D\cap\overline{B}^2(x_0,cr)$, then we say that D is a c-inner linearly locally connected domain, denoted by $D\in c-ILC$; (2) If for any $x_0\in R^2$ and $0<r<+\infty$, points in $D\setminus B^2(x_0,r)$ can be joined by curves in $D\setminus B^2(x_0,r/c)$, then we say that D is a c-outer linearly locally connected domain, denoted by $D\in c-OLC$.

D is called a linearly locally connected domain if $D\in c-ILC$ and $D\in c-OLC$ for some $c\ge 1$.

The following Example 1 shows that there exists a domain D which is not $c-ILC$ and $c-OLC$ at the same time for any $c\ge 1$.

Example 1 Let

$$\begin{aligned}
D_1&=\{(x_1,x_2)|x_1^2+(x_2-1)^2<1,x_1<0,x_2<1\},\\
D_2&=\{(x_1,x_2)|x_1^2+(x_2+1)^2<1,x_1<0,x_2>-1\},\\
D_3&=\{(x_1,x_2)|x_1\ge 0,-1<x_2<1\}
\end{aligned}$$

and

$$D=(D_1\cup D_2\cup D_3)\setminus\{(0,0)\}.$$

Then, the simple connected domain $D\subset R^2$ is not $c-ILC$ and $c-OLC$ simultaneously for any $c\ge 1$. In fact,

(I) For $x>0$, denote A the point $(-x,0)$, O the point $(0,0)$, $\mathrm{d}(A,\partial D)$ and $\mathrm{d}(A,O)$ the Euclidean distance from A to ∂D and O, respectively. Then,

$$\lim_{x\to 0}\frac{\mathrm{d}(A,O)}{\mathrm{d}(A,\partial D)}=\lim_{x\to 0}\frac{x}{\sqrt{1+x^2}-1}=+\infty.$$

Hence, for any $c\ge 1$, there exists a point $A(-x,0)$ such that $\mathrm{d}(A,O)>4c\mathrm{d}(A,\partial D)$. This concludes that there exist points in $D\cap\overline{B}^2(A,2\mathrm{d}(A,\partial D))$ which cannot be joined by curves in $D\cap\overline{B}^2(A,2c\mathrm{d}(A,\partial D))$, thus D is not $c-ILC$.

(II) For any $c\ge 1$, denote B the point $(3c^2,0)$, $r=2c^2$. It is obvious that there exist points in $D\setminus B^2(B,r)$, which cannot be joined by curves in $D\setminus B^2(B,r/c)$, thus D is not $c-OLC$.

The concept of linearly locally connected domains was first introduced by F.W. Gehring and J. Väisälä [14] in 1965 when they studied the properties of quasiconformal mappings in 3-space. Later, the concept was extensively used to study the quasidisks and the univalence of analytic functions, see [6–9, 11].

In this article, we shall clarify the relations between John disks and the following concepts that have not been studied up to the present:

(1) Inner and outer linearly locally connected domains;

(2) Quasiconformal mappings;

(3) Quasidisks.

We arrange this article as follows. In Section 2, we shall prove that a John disk must be an outer linearly locally connected domain, and the exterior of a John disk must be an inner linearly locally connected domain. In Section 3, we shall prove that a homeomorphism $f: R^2 \to R^2$ is a quasiconformal mapping if and only if $f(D)$ is a John disk for any John disk $D \subset R^2$. In Section 4, we shall prove that a bounded quasidisk must be a John disk, and construct an example to show that there exists an unbounded quasidisk, which is not a John disk.

2 John Disks and Linearly Locally Connected Domains

In this section, we shall prove the following two results:

Theorem 2.1 If $D \subset \overline{R}^2$ is a c-John disk, then $D \in (2c+2)-OLC$.

Proof Taking $b = 2c+2$. If $D \notin b-OLC$, then there exist $y_0 \in R^2$, $0 < r < +\infty$, and $x_1, x_2 \in D \setminus B^2(y_0, r)$, such that x_1 and x_2 cannot be joined by any curve in $D \setminus B^2(y_0, r/b)$.

Since D is a c-John disk, there exist $x_0 \in D$ and rectifiable curves $\gamma_j \subset D$, such that γ_j joins x_j to x_0 with $l(\gamma_j(x_j, x)) \le c\mathrm{d}(x, \partial D)$ for all $x \in \gamma_j$, $j = 1, 2$. If $\gamma = \gamma_1 \cup \gamma_2$, then $\gamma \subset D$, γ joins x_1 and x_2, and $\gamma \not\subseteq D \setminus B^2(y_0, r/b)$. If $y \in \gamma \cap \partial B^2(y_0, r/b)$, then $y \in \gamma_j$, $j = 1$ or 2, and

$$\mathrm{d}(y, \partial D) \ge \frac{1}{c} l(\gamma_j(x_j, y)) \ge \frac{1}{c}|x_j - y| \ge \frac{1}{c}(1 - \frac{1}{b})r,$$

this implies

$$B^2(y, \frac{1}{c}(1 - \frac{1}{b})r) \subseteq D, \tag{1}$$

but

$$\overline{B}^2(y_0, r/b) \not\subseteq D. \tag{2}$$

The above (1), (2), and the triangular inequality yield

$$\frac{1}{c}(1 - \frac{1}{b})r \le \frac{r}{b} + \frac{r}{b},$$

$$b \le 2c + 1, \tag{3}$$

which contradicts with $b = 2c+2$, hence $D \in (2c+2) - OLC$.

Theorem 2.2 If $D^* = \overline{R}^2 \setminus \overline{D}$ is a c_0-John disk, then $D \in (16c_0 + 21) - ILC$.

Proof Take $\delta = 8c_0 + 10$. For any $u \in R^2$, $s > 0$, take $z_1, z_2 \in D \cap \overline{B}^2(u, s)$, $z_1 \ne z_2$. Denote $z = \frac{1}{2}(z_1 + z_2)$ and $r = |z_1 - z_2|$. We first prove that z_1, z_2 must be in the same component of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$.

If z_1, z_2 belong to different components of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$, then z_1, z_2 must be in different components of $\overline{B}^2(z, \frac{1}{2}r) \setminus D^*$. Let β be the line segment joining z_1 and z_2, then β contains a subcurve $\alpha \subset D^*$, α dividing D^* into D_1 and D_2, and $\mathrm{dia}(D_j) \ge \frac{1}{2}r(\delta - 1)$, $j = 1, 2$. This yields

$$\min_{j=1,2} \mathrm{dia}(D_j) \ge \frac{1}{2}r(\delta - 1). \tag{4}$$

For any $x \in \alpha$, if $D_1 \nsubseteq B^2(x,(2c_0+2)\mathrm{dia}(\alpha))$ and $D_2 \nsubseteq B^2(x,(2c_0+2)\mathrm{dia}(\alpha))$, then take

$$x_j \in D_j \setminus \overline{B}^2(x,(2c_0+2)\mathrm{dia}(\alpha)), \qquad j=1,2.$$

Since D^* is a c_0-John disk, there exist a point $x_0 \in D^*$ and rectifiable curves $\gamma_j \subset D^*(j=1,2)$ joining x_j and x_0, such that for any $z \in \gamma_j$,

$$l(\gamma_j(x_j,z)) \le c_0 \mathrm{d}(z,\partial D^*), \qquad j=1,2.$$

If take $\gamma = \gamma_1 \cup \gamma_2$, then for any $w \in \gamma$ we have

$$\min_{j=1,2} l(\gamma(x_j,w)) \le c_0 \mathrm{d}(w,\partial D^*). \tag{5}$$

If $y \in \gamma \cap \partial B^2(x,\mathrm{dia}(\alpha))$, then we can get

$$\begin{cases} |y-x_j| \ge (2c_0+1)\mathrm{dia}(\alpha), \\ \min\limits_{j=1,2} l(\gamma(x_j,y)) \le c_0 \mathrm{d}(y,\partial D^*), \end{cases} \qquad j=1,2. \tag{6}$$

Which implies

$$B^2\left(y, \frac{2c_0+1}{c_0}\mathrm{dia}(\alpha)\right) \subseteq D^*, \tag{7}$$

but

$$B^2(x,\mathrm{dia}(\alpha)) \nsubseteq D^*. \tag{8}$$

(7), (8), and the triangular inequality imply

$$\frac{2c_0+1}{c_0}\mathrm{dia}(\alpha) < \mathrm{dia}(\alpha) + \mathrm{dia}(\alpha),$$

so

$$\mathrm{dia}(\alpha) < 0,$$

which is obviously impossible. Hence $D_1 \subseteq \overline{B}^2(x,(2c_0+2)\mathrm{dia}(\alpha))$ or $D_2 \subseteq \overline{B}^2(x,(2c_0+2)\mathrm{dia}(\alpha))$, and we can obtain

$$\min_{j=1,2} \mathrm{dia}(D_j) \le 2(2c_0+2)\mathrm{dia}(\alpha). \tag{9}$$

(4), (9), and $\mathrm{dia}(\alpha) \le r$ imply

$$\delta \le 8c_0 + 9,$$

which contradicts with $\delta = 8c_0+10$. Hence z_1, z_2 must be in the same component of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$, and there exists a rectifiable curve $\gamma \subseteq D$ joining z_1 and z_2, which satisfies

$$\mathrm{dia}(\gamma) \le \delta r = \delta |z_1 - z_2| \le 2\delta s. \tag{10}$$

This implies

$$\begin{aligned} \gamma &\subseteq D \cap \overline{B}^2(u, s+\mathrm{dia}(\gamma)) \subseteq D \cap \overline{B}^2(u,(2\delta+1)s) \\ &= D \cap \overline{B}^2(u,(16c_0+21)s). \end{aligned}$$

Hence $D \in (16c_0+21) - ILC$, this completes the proof of Theorem 2.2.

3 John Disks and Quasiconformal Mappings

To prove our main Theorem 3.1 mentioned below, we introduce and establish the following three lemmas first.

Lemma 3.1 [18] A Jordan domain $D \subseteq \overline{R}^2$ is a John disk if and only if there exist a constant $b \geq 1$ and a point $x_0 \in D$ such that, for any $x \in D$, there exists a curve $\beta \subseteq D$, which joins x and x_0, satisfying $\mathrm{dia}(\beta(x,y)) \leq b\mathrm{d}(y, \partial D)$ for any $y \in \beta$, where $\mathrm{dia}(\beta(x,y))$ denotes the Euclidean diameter of the subcurve β between x and y.

Lemma 3.2 Suppose that $f: R^2 \to R^2$ is a K-quasiconformal mapping, and $x \in R^2$. If $0 < r_1 \leq r_2 < +\infty$, then

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \leq c\left(\frac{r_2}{r_1}\right)^K, \tag{11}$$

where $c = c(K)$ is a constant, which depends only on K.

Proof Since $f: R^2 \to R^2$ is a K-quasiconformal mapping, for any $x \in R^2$ and $0 < r < +\infty$, making use of [21, p.79], we know

$$\frac{L(x,f,r)}{l(x,f,r)} \leq c' = c'(K), \tag{12}$$

where c' is a constant, which depends only on K.

For $0 < r_1 \leq r_2 < +\infty$, from (12) we can get

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \leq c'^2 \frac{l(x,f,r_2)}{L(x,f,r_1)}. \tag{13}$$

Without loss of generality, we may assume that $l(x,f,r_2) > L(x,f,r_1)$. Let Γ be the family of curves joining $S^1(x,r_1)$ and $S^1(x,r_2)$ in $\{z : r_1 < |z-x| < r_2\}$, $M(\Gamma)$ denotes the modulus of Γ. The comparison and monotonicity principles of modulus in [21] yield

$$\frac{2\pi}{\log\frac{l(x,f,r_2)}{L(x,f,r_1)}} \geq M(f(\Gamma)) \geq \frac{1}{K}M(\Gamma) = \frac{2\pi}{K\log\frac{r_2}{r_1}}. \tag{14}$$

Combining (13) and (14), we get

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \leq c'^2\left(\frac{r_2}{r_1}\right)^K = c\left(\frac{r_2}{r_1}\right)^K.$$

Lemma 3.3 [15] Let $f: R^2 \to R^2$ be a homeomorphism. If there exists a constant $c > 0$ such that

$$\left[\mathrm{dia}(f(B^2(x,r)))\right]^2 \leq c \cdot m[f(B^2(x,r))] \tag{15}$$

for any $x \in R^2$ and $r > 0$, then f is a quasiconformal mapping. Here $m[f(B^2(x,r))]$ denotes the 2-dimensional Lebesgue measure of $f(B^2(x,r))$.

Now, we can prove the following:

Theorem 3.1 A homeomorphism $f: R^2 \to R^2$ is a quasiconformal mapping if and only if $f(D)$ is a John disk for any John disk $D \subseteq R^2$.

Proof "The necessity." Suppose that f is a K-quasiconformal mapping, $D \subseteq R^2$ is a John disk. We know that there exists a constant $b \geq 1$ and $x_0 \in D$ such that D satisfies the equivalent definition for John disks in Lemma 3.1.

For any $y_1 \in f(D)$, taking $x_1 = f^{-1}(y_1) \in D$, Lemma 3.1 implies that there exists a curve $\gamma \subseteq D$ joining x_1 to x_0 and

$$\gamma(x_1, z) \subseteq \overline{B}^2(z, b\mathrm{d}(z, \partial D)), \qquad \text{for all} \qquad z \in \gamma.$$

If $\gamma' = f(\gamma)$, then $\gamma' \subseteq f(D)$ and γ' joins y_1 to $f(x_0)$. For any $y \in \gamma'$, taking $x = f^{-1}(y) \in \gamma$, and let $d = \mathrm{d}(x, \partial D)$, $l_1 = l(x, f, d)$, and $L_2 = L(x, f, bd)$.

Since $B^2(x, d) \subseteq D$ and $\gamma(x_1, x) \subseteq \overline{B}^2(x, bd)$, hence we have

$$B^2(f(x), l_1) = B^2(y, l_1) \subset f(D), \qquad \gamma'(y_1, y) \subseteq \overline{B}^2(y, L_2). \tag{16}$$

(16) implies

$$l_1 \leq \mathrm{d}(y, \partial f(D)) \tag{17}$$

and Lemma 3.2 implies

$$L_2 \leq cb^K l_1. \tag{18}$$

Combining (16), (17), and (18), we can obtain

$$\gamma'(y_1, y) \subseteq \overline{B}^2 \left(y, cb^K \mathrm{d}(y, \partial f(D))\right).$$

This yields $\mathrm{dia}(\gamma'(y_1, y)) \leq 2cb^K \mathrm{d}(y, \partial f(D))$, hence $f(D)$ is a John disk by Lemma 3.1.

"Sufficiency." For any $x \in R^2$ and $r > 0$, choosing $y \in B^2(x, r)$ such that

$$\mathrm{dia}[f(B^2(x, r))] \leq 3|f(x) - f(y)|. \tag{19}$$

Since $B^2(x, r)$ is a 1-John disk, the condition of Theorem 3.1 implies that there exists a constant $b \geq 1$ such that $f(B^2(x, r))$ is a b-John disk, so there exist $z_0 \in f(B^2(x, r))$ and rectifiable curves γ_1 and γ_2 in $f(B^2(x, r))$, such that γ_1 joins $f(x)$ to z_0, which satisfies

$$l(\gamma_1(f(x), z)) \leq b\mathrm{d}(z, \partial(f(B^2(x, r)))), \quad \text{for all} \quad z \in \gamma_1, \tag{20}$$

and γ_2 joins $f(y)$ to z_0 which satisfies

$$l(\gamma_2(f(y), z)) \leq b\mathrm{d}(z, \partial(f(B^2(x, r)))), \quad \text{for all} \quad z \in \gamma_2. \tag{21}$$

(20),(21), and triangular inequality imply

$$\begin{aligned} |f(x) - f(y)| &\leq l(\gamma_1) + l(\gamma_2) \\ &= l(\gamma_1(f(x), z_0)) + l(\gamma_2(f(y), z_0)) \\ &\leq 2b\mathrm{d}(z_0, \partial(f(B^2(x, r)))). \end{aligned} \tag{22}$$

Combining (19) and (22), we get

$$\mathrm{d}(z_0, \partial(f(B^2(x, r)))) \geq \frac{|f(x) - f(y)|}{2b} \geq \frac{\mathrm{dia}[f(B^2(x, r))]}{6b}. \tag{23}$$

This gives

$$\begin{aligned} &B^2 \left(z_0, \frac{\mathrm{dia}[f(B^2(x, r))]}{6b}\right) \subseteq f(B^2(x, r)), \\ &\pi \left(\frac{\mathrm{dia}[f(B^2(x, r))]}{6b}\right)^2 \leq m(f(B^2(x, r))), \\ &\left(\mathrm{dia}[f(B^2(x, r))]\right)^2 \leq \frac{(6b)^2}{\pi} m(f(B^2(x, r))). \end{aligned} \tag{24}$$

(24) and Lemma 3.3 imply that f is a quasiconformal mapping.

4 John Disks and Quasidisks

Let D be a proper subdomain of R^2, for each pair of points $x_1, x_2 \in D$, we set

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \mathrm{d}(x, \partial D)^{-1} \mathrm{d}s, \tag{25}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. We call k_D the quasi-hyperbolic metric in D. A rectifiable arc $\gamma \subseteq D$ is said to be a quasi-hyperbolic geodesic if

$$k_D(y_1, y_2) = \int_{\gamma(y_1, y_2)} \mathrm{d}(x, \partial D)^{-1} \mathrm{d}s \tag{26}$$

for each pair of points $y_1, y_2 \in \gamma$. Obviously, each subarc of a quasi-hyperbolic geodesic is again a geodesic. From Lemma 1 in [12] we know that each pair of points $x_1, x_2 \in D$ can be joined to a quasi-hyperbolic geodesic γ in D.

From Lemma 2.1 in [13] it follows that

$$\begin{cases} 0 \left| \log \dfrac{\mathrm{d}(x_1, \partial D)}{\mathrm{d}(x_2, \partial D)} \right| \leq k_D(x_1, x_2), \\ \log \left(\dfrac{|x_1 - x_2|}{\mathrm{d}(x_j, \partial D)} + 1 \right) \leq k_D(x_1, x_2), \quad j = 1, 2 \end{cases} \tag{27}$$

for all $x_1, x_2 \in D$. Hence

$$j_D(x_1, x_2) \leq k_D(x_1, x_2), \tag{28}$$

where

$$j_D(x_1, x_2) = \frac{1}{2} \log \left(\frac{|x_1 - x_2|}{\mathrm{d}(x_1, \partial D)} + 1 \right) \left(\frac{|x_1 - x_2|}{\mathrm{d}(x_2, \partial D)} + 1 \right).$$

F.W. Gehring and K. Hag proved the following result in [10].

Lemma 4.1 [10] A simply connected proper subdomain D of R^2 is a quasidisk if and only if there exists a constant c such that

$$k_D(x_1, x_2) \leq c j_D(x_1, x_2) \tag{29}$$

for all $x_1, x_2 \in D$.

The main result of this section is the following theorem.

Theorem 4.1 If $D \subseteq R^2$ is a bounded quasidisk, then D is a John disk.

Proof Since D is a bounded domain, there exists a point $x_0 \in D$ such that $\mathrm{d}(x_0, \partial D) = \max\limits_{x \in D} \mathrm{d}(x, \partial D)$. For any $x_1 \in D$, let γ be the quasi-hyperbolic geodesic in D which joins x_1 and x_0.

Define $y_1, y_2, \cdots, y_m, y_{m+1} \in \gamma$ inductively as follows. Put $y_1 = x_1$, and if $y_j \in \gamma (j \geq 1)$ is defined, let $d_j = \mathrm{d}(y_j, \partial D)$, then y_{j+1} is the first point of $\gamma(y_j, x_0)$ for which $d_{j+1} = \mathrm{d}(y_{j+1}, \partial D) = 2d_j$ as we traverse γ from y_j towards x_0 for $\mathrm{d}(x_0, \partial D) > 2d_j$ and $y_{j+1} = x_0 (j = m)$ for $\mathrm{d}(x_0, \partial D) \leq 2d_j$.

Let $\gamma_j = \gamma(y_j, y_{j+1})$ and $l_j = l(\gamma_j)$. If $x \in \gamma_j, j = 1, 2, \cdots, m-1$, then

$$\mathrm{d}(x, \partial D) \leq 2d_j. \tag{30}$$

If $j = m$, then

$$\mathrm{d}(x, \partial D) \le \mathrm{d}(x_0, \partial D) \le 2d_m. \tag{31}$$

(30) and (31) imply

$$\frac{l_j}{d_j} \le 2\int_{\gamma_j} \frac{\mathrm{d}s}{\mathrm{d}(x, \partial D)} = 2k_D(y_j, y_{j+1}) \tag{32}$$

for $j = 1, 2, \cdots, m$.

Since D is a quasidisk, there exists a constant c such that

$$\begin{aligned} k_D(y_j, y_{j+1}) \le c j_D(y_j, y_{j+1}) &= \frac{c}{2}\log\left(1 + \frac{|y_j - y_{j+1}|}{\mathrm{d}(y_j, \partial D)}\right)\left(1 + \frac{|y_j - y_{j+1}|}{\mathrm{d}(y_{j+1}, \partial D)}\right) \\ &\le c\log\left(1 + \frac{|y_j - y_{j+1}|}{\mathrm{d}(y_j, \partial D)}\right) \le c\log\left(1 + \frac{l_j}{d_j}\right) \le c\left(\frac{l_j}{d_j}\right)^{\frac{1}{2}} \end{aligned} \tag{33}$$

by Lemma 4.1.

Making use of (32) and (33), we can get

$$l_j \le 4c^2 d_j, \qquad j = 1, 2, \cdots, m. \tag{34}$$

For any $x \in \gamma$, there exists $j \le m$ such that $x \in \gamma_j$, and (27) implies

$$\log\frac{d_j}{\mathrm{d}(x, \partial D)} \le k_D(y_j, x) \le k_D(y_j, y_{j+1}). \tag{35}$$

Combining (33), (34), and (35), we have

$$d_j \le e^{2c^2}\mathrm{d}(x, \partial D). \tag{36}$$

From (34), (36), and the definition of y_j, we can obtain

$$\begin{aligned} l(\gamma(x_1, x)) &\le \sum_{i=1}^{j} l_i \le 4c^2\sum_{i=1}^{j} d_i \le 4c^2(2^{1-j}d_j + 2^{2-j}d_j + \cdots + d_j) \\ &< 8c^2 d_j \le 8c^2 e^{2c^2}\mathrm{d}(x, \partial D), \end{aligned} \tag{37}$$

hence D is a John disk.

Next, we construct an example to show that there exists an unbounded quasidisk which is not a John disk.

Example 2 Let $D = \{z : \mathrm{Im}(z) > 0\}$ be the upper half-plane, it is obvious that D is a quasidisk. We shall prove that D is not a John disk.

For any $x_0 \in D$ and $b \ge 1$, let $a = Im(x_0) = \mathrm{d}(x_0, \partial D)$, $x = \mathrm{Re}(x_0) + [2a(1+b)+1]i \in D$, and take any rectifiable curve $\gamma \subseteq D$ joining x and x_0. If taking $y \in \gamma \cap \partial B^2(x_0, a)$, then

$$b\mathrm{d}(y, \partial D) \le 2ab,$$

but

$$l(\gamma(x, y)) \ge [2a(1+b)+1] - 2a = 2ab + 1 > 2ab.$$

Hence $l(\gamma(x, y)) > b\mathrm{d}(y, \partial D)$, this shows that D is not a John disk.

Finally, we give the following corollary directly from Theorem 2.1, Theorem 2.2, and the result in [11].

Corollary 4.1 Let $D \subseteq \overline{R}^2$ be a Jordan domain, if both D and $D^* = \overline{R}^2 \setminus \overline{D}$ are John disks, then D is a quasidisk.

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